Patterns in Primes and Bunyakovsky's Conjecture

Bruno Eaves, Ivan Noden, Ioannis Pantelidakis

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- Are there any more?

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- Is p prime infinitely often?

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- If we stick to the convention that prime numbers are positive then since polynomials like $-x^2 + 4$ are only positive for a finite number of natural numbers it can clearly only be prime finitely many times.

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- Consider a polynomial like $2x^2$.
- This will only be prime for $x = 1$ as every other value will be divisible by 2.
- To account for this, we can discount polynomials f where $\gcd(f(1), f(2), \ldots) \neq 1$.
- If this were the case, there must be some prime p such that p divides $f(n)$ for all natural numbers n.
- However, $f(n)$ can only equal p a finite number of times.
- So every other value of $f(n)$ will have p as a proper divisor.

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- Consider $f(x) = 2x^4 - 7x^2 + 3 = (x^2 - 3)(2x^2 - 1)$.
- Note then that $f(n)$ can be written as a product of two integers.
- One of these factors can be ± 1 only finitely many times.
- So $f(n)$ can be prime only finitely many times.
- Clearly this is true whenever f is reducible.

Bunyakovsky's Conjecture

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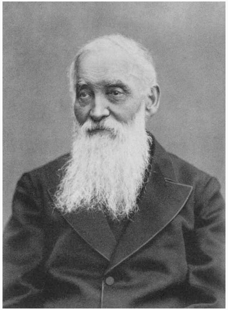
Let $f \in \mathbb{Z}[x]$ be such that

- f is irreducible over $\mathbb{Z}$
- The leading coefficient of f is positive
- $\gcd(f(1), f(2), f(3), \ldots) = 1$

then there are infinitely many $n \in \mathbb{N}$ such that $f(n)$ is prime.



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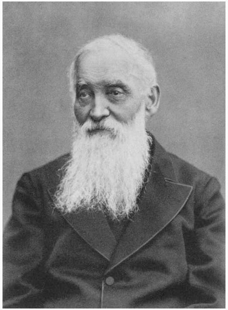

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We believe this to be true but we have only been able to prove so when f is linear. This case is known as Dirichlet's theorem.

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- See that this case is equivalent to asking if there are infinitely many primes congruent to a mod m
- We can prove some cases using a method similar to Euclid's.
- How can we do it in general?

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- Assume it is true, so that, for all coprime integers m and a, there exists a prime congruent to a mod m.
- Fix a pair of coprime integers m_0 and a_0
- There thus exists $x \in \mathbb{Z}$ such that $p = m_0x + a_0$ is prime.
- Note, p and m_0 are coprime so there exists a $y \in \mathbb{Z}$ such that $q = m_0y + p$ is prime.

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- But $q = m_0 y + p = m_0(x + y) + a_0$.
- So, q is another prime congruent to a_0 mod m_0 but different from p.
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- So, q is another prime congruent to a_0 mod m_0 but different from p.
- It follows by induction that there must be infinitely many primes congruent to a_0 mod m_0.
- Thus, if $f(x) = mx + a$ is prime once then it is prime infinitely often.

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- We can write

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- Taking logarithms and manipulating we get

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- In a similar way, we can show that the series

$$\sum_{\substack{p \equiv a \bmod m \\ p \text{ prime}}} \frac{1}{p^s}$$

  diverges as $s \to 1$, proving Dirichlet's theorem.

Green-Tao Theorem

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- We expect the last prime in the sequence should be around

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where $\gamma = \lim_{n\to\infty}(-\log n + 1 + \frac{1}{2} + \cdots + \frac{1}{n})$

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  where $\gamma = \lim_{n\to\infty}(-\log n + 1 + \frac{1}{2} + \cdots + \frac{1}{n})$
- So if f were to beat $x^2 + x + 41$ and $f(40)$ was prime then $f(40)$ would be around 4.9×10^{29}
- For comparison, there are roughly 7×10^{27} atoms in a human body

Green-Tao Theorem

One final remark:

- Suppose $mx + a$ is prime for the first k^d values
- Then $mx^d + a$ is prime for the first k values

Schinzel's Conjecture

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Let $\{f_1, f_2, \ldots, f_k\}$ be a finite set of non-constant, irreducible polynomials over $\mathbb{Z}$ with positive leading coefficients such that there does not exist a prime p where p divides $f_1(n) \cdots f_k(n)$ for all $n \in \mathbb{N}$.

Then there are infinitely many $n \in \mathbb{Z}$ such that $f_1(n), \ldots, f_k(n)$ are all prime.

Andrzej Schinzel

This final condition prevents counter-examples such as $\{x, x+1\}$ where 2 always divides $x(x+1)$.

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Schinzel's conjecture generalises many pre-existing conjectures related to patterns in primes. In particular that there exist infinitely many pairs of

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- Sophie Germain primes: $\{x, 2x + 1\}$
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- Cousin primes: $\{x, x + 4\}$
- Twin primes: $\{x, x + 2\}$

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- The dream is to prove the twin prime conjecture, that is the case when $c = 2$.
- In 2013, Zhang proved there exists some $c < 70$ million for which the statement holds
- With some tweaking of Zhang's work, the bound was able to be reduced to 20 million
- Polymath8 reduced this further to 4680
- New methods by Tao and Maynard independently reduced this bound to 246

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Twin Prime Conjecture

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- Many primes we often encounter (3 and 5, 11 and 13) are twin primes
- The smallest pair greater than a billion is $1,000,000,007$ and $1,000,000,009$
- How often do they occur?

Hardy-Littlewood Conjecture

- By the prime number theorem, the number of primes less than n is asymptotic to

$$\frac{n}{\log n}$$

- So we may expect the number of primes p such that $p + 2$ is prime to be asymptotic to

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- This clearly can't be true as the same argument would apply to p and $p+1$

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- We introduce the constant

$$C = 2 \prod_{\substack{q \text{ prime} \\ q \geq 3}} \frac{1 - 2/q}{(1 - 1/q)^2} = 1.3203236316\ldots$$

which takes into account the probability that p and $p+2$ have common divisors.

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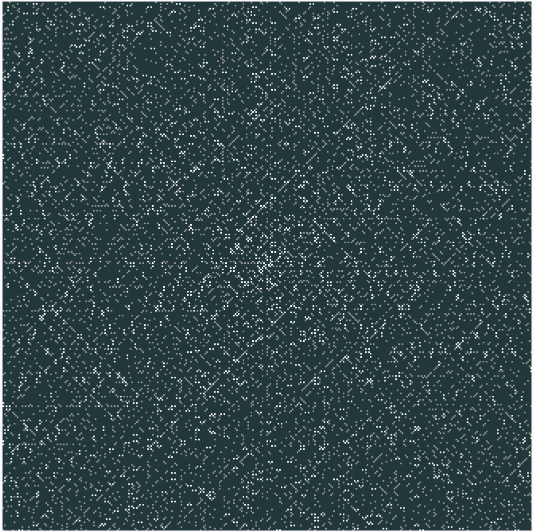
- There are 8 twin primes less than 100, our estimate precicts there to be roughly 6.225, a 22% error.
- There are $3, 424, 506$ twin primes less than a billion, our estimate predicts there to be roughly $3, 074, 425$, a 10% error.

Ulam Spiral



Conclusion

- Like Schinzel's, Bunyakovsky's and the Twin Prime Conjecture, the Hardy-Littlewood Conjecture seems almost certainly true yet we seem far from proving it
- It seems easy for us to recognise patterns among the primes but the truth behind them is far more elusive

Questions?